VECTOR QUANTIZATION AND SIGNAL COMPRESSION

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KLUWER ACADEMIC PUBLISHERS
Boston/Dordrecht/London

Best Available Copy

Distributors for North America:
Kluwer Academic Publishers
101 Philip Drive
Assinippi Park
Norwell, Massachusetts 02061 USA

Distributors for all other countries:
Kluwer Academic Publishers Group
Distribution Centre
Post Office Box 322
3300 AH Dordrecht, THE NETHERLANDS

Library of Congress Cataloging-in-Publication Data

Gersho, Allen.
Vector quantization and signal compression / by Allen Gersho, Robert M. Gray.
p. cm. -- (Kluwer international series in engineering and computer science ; SECS 159)
Includes bibliographical references and index.
ISBN 0-7923-9181-0
1. Signal processing--Digital techniques. 2. Data compression (Telecommunication) 3. Coding theory. I. Gray, Robert M., 1943- . II. Title. III. Series.
TK5102.5.G45 1991
621.382'2--dc20
91-28580
CIP

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Printed on acid-free paper.

Printed in the United States of America

Since there is an infinite variety of orthogonal matrices to choose from, all of which prevent magnification of quantizing noise, a natural approach is to find a matrix $\mathbf{T}$ which eliminates or nearly eliminates the correlation between the transform coefficients. We shall see that the effect of eliminating correlation is to reduce the product of the variances of the vector components, which in turn will reduce the average distortion attainable for the given bit quota when the optimal bit allocation under the high rate approximation of (8.3.1) is used.

8.6 Karhunen-Loeve Transform

In general, the components of $\mathbf{X}$ are correlated with one another. However, it is indeed possible to select an orthogonal matrix $\mathbf{T}$, for a given pdf describing $\mathbf{X}$, that will make $\mathbf{Y} = \mathbf{TX}$ have pairwise uncorrelated components. This choice of transform matrix that makes the linear transformation have this desirable property is called the *discrete time Karhunen-Loeve transform* or the *Hotelling transform* and is defined below.

Let $\mathbf{R}_X = E[\mathbf{XX}^t]$, denote the autocorrelation matrix of the input (column) vector $\mathbf{X}$. (If the vectors are complex this becomes $\mathbf{R}_X = E[\mathbf{XX}^*]$.) Let $\mathbf{u}_i$ denote the eigenvectors of $\mathbf{R}_X$ (normalized to unit norm) and λ_i the corresponding eigenvalues. Since any autocorrelation matrix is symmetric and nonnegative definite, there are k orthogonal eigenvectors and the corresponding eigenvalues are real and nonnegative. Without loss of generality we assume the indexing is such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

The Karhunen-Loeve transform matrix is then defined as $\mathbf{T} = \mathbf{U}^t$, where

$$\mathbf{U} = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k],$$

that is, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{R}_X$.

Then the autocorrelation matrix of $\mathbf{Y}$ is given by

$$\begin{aligned} \mathbf{R}_Y &= E[\mathbf{YY}^t] = E[\mathbf{U}^t\mathbf{XX}^t\mathbf{U}] \\ &= \mathbf{U}^t\mathbf{R}_X\mathbf{U} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k \end{bmatrix} \end{aligned}$$

Thus we see that the Karhunen-Loeve transform does indeed decorrelate the input vector. It also follows that the variances of the transform coefficients are the eigenvalues of the autocorrelation matrix R_X.

Suppose we apply optimal quantization to each transform coefficient. For optimal bit allocation, the mean square distortion resulting from transform coding with the Karhunen-Loeve transform is given by

$$D_{\mathrm{tc}} = kH\left(\prod_{i=1}^{k} \lambda_i\right)^{\frac{1}{k}} 2^{-2\bar{b}}$$

where H is the geometric mean of the quantization coefficients h_i for the normalized pdf's Y_i. It is important to note that H depends on the transform matrix $\mathbf{T}$ and on the joint pdf of the input vector $\mathbf{X}$. In the special case where $\mathbf{X}$ is Gaussian, each Y_i is also Gaussian so that $H = h_g$, the quantization coefficient for the Gaussian pdf obtained by applying (8.2.3) for the Gaussian case and given by (8.2.4). We have used the fact that the variance of the coefficient Y_i is λ_i. Since the determinant of a matrix is the product of its eigenvalues, we have

$$D_{\mathrm{tc}} = kH(\det \mathbf{R}_X)^{\frac{1}{k}} 2^{-2\bar{b}}. \tag{8.6.1}$$

We have not yet shown that the Karhunen-Loeve transform is indeed the best possible transform for minimizing the overall distortion D_{tc} for a given bit allocation. This will be proved under the assumptions that the input variables are Gaussian, that optimal mean squared error quantization of the transform coefficients is performed, and that the optimal allocation result (8.3.3) (based on the high resolution assumption) holds.

Before proceeding, we note the following bound on the determinant of an autocorrelation matrix that will be useful later.

Covariance Determinant Bound

For any autocorrelation (covariance) matrix $\mathbf{R}_v$ of a real-valued random vector $\mathbf{V}$ whose components have zero mean and variances $E[V_i^2] = \sigma_i^2$, we have

$$\det \mathbf{R}_v \leq \prod_{i=1}^{k} \sigma_i^2.$$

Furthermore, equality holds if and only if $\mathbf{R}_v$ is a diagonal matrix.

As an illustration of this result, consider the case where $k = 2$. The determinant is then given by $\sigma_1^2\sigma_2^2(1-\rho^2)$ where ρ is the correlation coefficient. This clearly satisfies the bound. In this case, and in general, it can be shown that the more positively correlated the vector components are, the smaller is the determinant. It should also be noted that the determinant of an autocorrelation matrix is always nonnegative.